SOLUTIONS
MID-SEMESTER EXAM B-MATH-III YEAR (DIFFERENTIAL TOPOLOGY)
2015

Solution 1(i): As the map f is a local chart, f is immersion. Hence, for a point $(s,t) \in (0,2\pi) \times (0,2\pi)$, the total derivative $Df(s,t) : \mathbb{R}^2 \to \mathbb{R}^3$ is injective. The tangent space $T_{(s,t)}(T^2)$ is the image of the linear map $D_{(s,t)}f$. Therefore, $T_{(s,t)}(T^2)$ is generated by the set $\{\partial f/\partial s, \partial f/\partial t\}$, where $\partial f/\partial s = (-\sin s \cos t, -\sin s \sin t, \cos s)$ and $\partial f/\partial t = (-(2+\cos s)\sin t, (2+\cos s)\cos t, 0)$.

Solution 1(ii): The submanifold T^2 of $\mathbb{R}^3$ is covered by four smooth local charts f_1, f_2, f_3 and f_4 where $f_1 : (0,2\pi) \times (0,2\pi) \to \mathbb{R}^3$ is given by:

$$f_1(s,t) = ((2+\cos s)\cos t, (2+\cos s)\sin t, \sin s),$$

$f_2(s,t) : (0,2\pi) \times (0,2\pi) \to \mathbb{R}^3$ is given by:

$$f_2(s,t) = ((2-\cos s)\cos t, (2-\cos s)\sin t, -\sin s),$$

$f_3 : (0,2\pi) \times (0,2\pi) \to \mathbb{R}^3$ is given by:

$$f_3(s,t) = (-(2+\cos s)\cos t, -(2+\cos s)\sin t, \sin s),$$

and $f_4 : (0,2\pi) \times (0,2\pi) \to \mathbb{R}^3$ is given by:

$$f_4(s,t) = (-(2-\cos s)\cos t, -(2-\cos s)\sin t, -\sin s).$$

Therefore a point (x,y,z) is a critical point of g on T^2 if $(x,y,z) = f_i(s_i,t_i)$ where (s_i,t_i) is a critical point of $g \circ f_i$, $i = 1,2,3,4$. A point (s_1,t_1) is a critical point of $g \circ f_1$ if and only if the linear map $D_{(s_1,t_1)}(g \circ f_1)$ is a zero map. That means

$$\frac{\partial (g \circ f_1)}{\partial s}(s_1,t_1) = 0 = \frac{\partial (g \circ f_1)}{\partial t}(s_1,t_1).$$

This implies that (s_1,t_1) is a critical point of $g \circ f_1$ if and only if

$$-\sin s_1 \cos t_1 = -(2+\cos s_1)\sin t_1 = 0.$$

Solving the above equality, we see that $(-\pi,-\pi)$ is the critial point of the map $g \circ f_1$. Similarly, we can prove that $(-\pi,-\pi)$ is a critical point of $g \circ f_i$ for each $i = 2,3,4$. Hence, points

$$(-1,0,0), (-3,0,0), (1,0,0) \text{ and } (3,0,0)$$

are the four critical points of the map g on T^2.

Solution 2(i): Let M be of dimension m and N be of dimension n. Let $p \in M$ be an arbitary point. As f is a submersion, by submersion theorem, there is a coordinate neighborhoods (U_p, ϕ) of p in M and (V_p, ψ) of $f(p)$ in N such that $f(U_p) = V_p$ and the transition map $\psi \circ \phi^{-1} : \phi(U_p) \to \psi(V_p)$ is the standard projection

$$(x_1, x_2, \ldots, x_n, x_{n+1}, \ldots, x_m) \to (x_1, x_2, \ldots, x_n).$$

As the standard projection maps are open, the image $f(V_p)$ is open.

Now, if U is an open subset of M, then

$$U = \bigcup_{p \in M} (U \bigcap U_p).$$

Therefore,

$$f(U) = \bigcup_{p \in M} f(U \bigcap U_p).$$

As each $f(U \bigcap U_p)$ is open in V_p and hence, $f(U \bigcap U_p)$ is open in N. This implies that $f(U)$ is open in N. This proves that any submersion map is an open map.

Solution 2(ii): As f is an open map, the image $f(M)$ is an open subset of N. On the other hand, as M is compact, the image $f(M)$ is compact, and hence closed subset of N. Since N is connected, we have $N = f(M)$. Hence, f is surjective.

Solutions 3(i): Note that a matrix A has distinct eigenvalues if the characteristic polynomial $P(A)$ has no repeated roots. The polynomial $P(A)$ has no repeated roots if the resultant of $P(A)$ with its derivative $P'(A)$ is non-zero. As this condition is open condition we conclude that set of (2×2) matrices with real and distinct roots are open subsets of $M(2, \mathbb{R})$.

Solutions 3(ii): To prove this it is enough to prove that the derivative $D_{(A,x)}f$ is a non-zero linear map. Now by Submersion theorem, $f^{-1}(0)$ is a smooth 4-dimensional manifold.

Solutions 4(i): Let $x \in M$ be a local maximum or local minimum of the smooth function $f : M \to \mathbb{R}$. There is a coordinate neighborhood (U_x, ψ) of x in M such that $\psi(x)$ is maximum or minimum of the function $f \circ \psi^{-1}$. Consider the basis $\{\partial/\partial x_1, \partial/\partial x_2, \ldots, \partial/\partial x_n\}$ of the tangent space $T_{\psi(x)}\psi(U_x)$. Corresponding to each basis vector $\partial/\partial x_i$, let $\alpha_i : V_i \to \psi(U_x)$ be a smooth map such that V_i is an open neighborhood of $0 \in \mathbb{R}$ with $\alpha_i(0) = \psi(x)$ and the derivative $\alpha_i'(0) = \partial/\partial x_i$. Now the point 0 is a local maximum or a local minumum of the function $f \circ \psi^{-1} \circ \alpha_i : V_i \to \mathbb{R}$. By the calculus on one variable, the derivation $(f \circ \psi^{-1} \circ \alpha_i)'(0) = 0$. This implies that $D_{\psi(x)}(f \circ \psi^{-1})(\partial/\partial x_i) = 0$. Hence, the linear map $D_{\psi(x)}(f \circ \psi^{-1})$ is a zero map. This implies that the derivative $D_x f : T_x M \to \mathbb{R}$ is a zero map.

Solution 4(ii): Consider the function $f_1 : M \to \mathbb{R}$ given by $f_1(x) = ||x||^2$. The derivation $D_x f_1 : T_x M \subset \mathbb{R}^n \to \mathbb{R}$ is given by $D_x f_1(v) = 2x \cdot v$, where $x \cdot v$ is the standard inner product on $\mathbb{R}^n$. Note that a local extremum of the function f is also a local extremum of the function f_1. Suppose x is a local extremum of f, then the derivation $D_x f_1$ is a zero map. This means that $x \cdot v = 0$ for all $v \in T_x M$. Hence the vector $x \in \mathbb{R}^n$ is normal to the tangent space $T_x M \subset \mathbb{R}^n$. As $D_x g$ is also normal to $T_x M$, and $T_x M$ is $(n-1)$ dimensional subspace of $\mathbb{R}^n$, we conclude that $D_x g$ and x are scalar multiple of each other.